ELSEVIER Journal of Pure and Applied Algebra 108 (1996) 231–240

JOURNAL OF PURE AND APPLIED ALGEBRA

An effective version of Pólya's theorem on positive definite forms

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Communicated by M.-F. Roy, received 28 August 1994

Abstract

Given a real homogeneous polynomial F, strictly positive in the non-negative orthant, Pólya's theorem says that for a sufficiently large exponent p the coefficients of $F(x_1,\dots,x_n)\cdot(x_1+\cdots+x_n)^p$ are strictly positive. The smallest such p will be called the Pólya exponent of F. We present a new proof for Pólya's result, which allows us to obtain an explicit upper bound on the Pólya exponent when F has rational coefficients. An algorithm to obtain reasonably good bounds for specific instances is also derived.

Pólya's theorem has appeared before in constructive solutions of Hilbert's 17th problem for positive definite forms [4]. We also present a different procedure to do this kind of construction.

1. Introduction

In 1928 Pólya [7] proved the following theorem (see also [5]):

Theorem 1.1 (Pólya). *Let $F(x_1,\dots,x_n)$ be a real homogeneous polynomial which is positive in $x_i \geq 0$, $\sum x_i > 0$. Then, for a sufficiently large integer p, the product*

$$F(x_1,\dots,x_n)\cdot(x_1+\cdots+x_n)^p$$

has all its coefficients strictly positive.

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[1] Partially supported by a doctoral fellowship of the National University of Mexico, the National Science Foundation, and the David and Lucile Packard Foundation

[2] Partially supported by DGICyT PB 92/0498-C02 and the David and Lucile Packard Foundation.

SSDI 0022-4049(95)00042-9

The smallest exponent p that satisfies the properties of the theorem will be called the *Pólya exponent* of F. Our purpose is to show an elementary derivation for an upper bound of the Pólya exponent. Using an effective Łojasiewicz inequality for the case of rational coefficients [10], this upper bound can be written in terms of the degree, the number of variables and the size of the coefficients of F. This is done in the following theorem.

Theorem 1.2. *Let $F(x_1,\ldots,x_n)$ be a real homogeneous polynomial of degree d whose coefficients are bounded in absolute value by $l \geq 2$. Suppose that F is strictly positive in the non-negative orthant (minus the origin). Denote by λ the minimum of F in the unit simplex $\Delta = \{\sum_{i=1}^{n} x_i = 1,\ x_i \geq 0\ \forall i\}$. Call D the maximum of $d+1$ and $n+1$. Under these assumptions:*

1. *If F has integer coefficients, then $1/\lambda$ is bounded above by $l^{D^{O(n)}}$.*
2. *For any integer p greater than $(\ln d^2 + dn)/\lambda$, the product*

$$F(x_1,\ldots,x_n)\cdot(x_1+\cdots+x_n)^p$$

has all its coefficients strictly positive.

We remark that recent work by Reznick (see [8]) contains results similar to part two of our theorem. Observe that Theorem 1.1. implies that F can be written in the form $F = G/H$, where G and H are homogeneous polynomials with only positive coefficients. This is a necessary and sufficient condition for F to be strictly positive in the non-negative orthant. In a similar way, Artin decomposition of a polynomial as a quotient of two sums of squares is necessary and sufficient to guarantee positive semidefiniteness in $\mathbb{R}^n$. Habicht [4] found a way to construct an Artin decomposition of a positive definite form using Pólya's theorem. In Section 3 we present a new method to do this.

Let us finally indicate that a slightly more general version of Pólya's theorem appears in the theory of Geometric Design (see Theorem 1.3 in [3]) in connection with the approximation of polynomial functions in a simplicial region. The generalization comes from the fact that the convergence result in the proof of Lemma 2.1 is still true for F not necessarily positive. This implies that, for large p, the coefficients of the polynomials $(x_1+\cdots+x_n)^p F(x_1,\ldots,x_n)$ approximate $F(x_1,\ldots,x_n)$ (up to a normalization) at some test points in the simplex $\Delta = \{\sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \quad \forall i\}$.

2. Proof of the main result

We will first present some notation. We will abbreviate $F(x_1,\ldots,x_n)$ by $F(X)$. The polynomial $F(X)$ can be written as a difference $F_+(X) - F_-(X)$ where the polynomials $F_+(X)$ and $F_-(X)$ have only positive coefficients. We use $\sum X$ to denote $x_1 + x_2 + \cdots + x_n$, and $X + d$ to abbreviate $(x_1 + d,\ldots,x_n + d)$. Finally $X > 0$ will indicate that $x_i > 0$ for $i = 1,\ldots,n$.

Lemma 2.1. *Let $F(X)$ be a real homogeneous polynomial of degree d, strictly positive in the non-negative orthant (minus the origin).*

1. *The semialgebraic region $G := \{X : F_+(X) - F_-(X+d) \le 0, X \ge 0\}$ is bounded.*
2. *For any p greater than or equal to dn plus the maximum of $\sum X$ on the region G, the product $F(X) \cdot (\sum X)^p$ has all its coefficients strictly positive.*

Proof. Observe that the part of largest degree of $F_*(X) = F_+(X) - F_-(X+d)$ is the polynomial $F(X)$ and the remaining terms have negative coefficients. Hence for each point Q in the simplex $\Delta = \{\sum X = 1, X \ge 0\}$ the univariate polynomial $H_Q(\lambda) = F_*(\lambda Q)$ has a positive leading term and the rest of its terms are negative. Call $r(Q)$ the only positive real root of $H_Q(\lambda)$. The function $r(Q)$ is continuous in Δ which is compact and thus attains a maximum. This finishes the proof of part (1).

For the proof of part (2), let $F(X) = \sum c_V X^V$, and $F(X)(\sum X)^p = \sum C_U X^U$, where $V = (v_1, \dots, v_n)$, $\sum v_i = d$ and $U = (u_1, \dots, u_n)$, $\sum u_i = p + d$. Then the coefficient C_U equals

$$\sum_{\Sigma v_i = d} c_V \frac{p!}{(u_1 - v_1)! \cdots (u_n - v_n)!} = \sum c_V P_{U,V}.$$

If $u_i > d$ for $i = 1, \dots, n$ then it is easy to see that the following two inequalities are satisfied (note tat $0 \le v_i \le d$ for all i):

$$\frac{p!}{u_1! \cdots u_n!} u_1^{v_1} \dots u_n^{v_n} \ge P_{U,V} \ge \frac{p!}{u_1! \cdots u_n!} (u_1 - d)^{v_1} \dots (u_n - d)^{v_n}.$$

Using one of these inequalities for each $P_{U,V}$ depending on the sign of c_V we get

$$\frac{u_1! \cdots u_n! C_U}{p!} \ge F_+(u_1 - d, \dots, u_n - d) - F_-(u_1, \dots, u_n).$$

Otherwise, one of the u_i is smaller than or equal to d. Without loss of generality assume $u_1, \dots, u_k > d \ge u_{k+1}, \dots, u_n$. In this case we have another pair of inequalities:

$$\frac{p!}{u_1! \cdots u_n!} (u_1)^{v_1} \cdots (u_k)^{v_k} d^{v_{k+1}} \dots d^{v_n} \ge P_{U,V},$$

$$P_{U,V} \ge \frac{p!}{u_1! \cdots u_n!} (u_1 - d)^{v_1} \cdots (u_k - d)^{v_k} 0^{v_{k+1}} \cdots 0^{v_n},$$

where 0^0 is taken to be 1 if $v_i = 0$ for some $i > k$. In the same way as before we conclude that

$$\frac{u_1! \cdots u_n! C_U}{p!} \ge F_+(u_1 - d, \dots, u_k - d, 0, \dots, 0) - F_-(u_1, \dots, u_k, d, \dots, d).$$

In both cases we obtain $(u_1! \cdots u_n! C_U / p!) \ge F_+(x_1, \dots, x_n) - F_-(x_1 + d, \dots, x_n + d)$ for certain $x_1, \dots, x_n$ with $\sum x_i > p - dn$. Using the assumption on p, we have $F_+(x_1, \dots, x_n) - F_-(x_1 + d, \dots, x_n + d) > 0$ and thus the coefficient C_U is positive. □

For the proof of Theorem 1.2 we want to give a procedure to find the maximum of the linear form $\sum X$ inside the region $G = \{X \in \mathbb{R}^n \mid F_+(X) - F_-(X+d) \le 0, X \ge 0\}$.

We will also derive a theoretical bound for this maximum using an effective Łojasiewicz inequality. The following statement is the quantifier free case of Lemma 5 in [10] (see Ch. 2 of [1] for general information on Łojasiewicz inequalities).

Lemma 2.2 (Solernó). *Let $V \subset \mathbb{R}^n$ be a non-empty and closed semialgebraic set and let $f : V \to R$ be a continuous semialgebraic function. Assume that both V and the graph of f are defined by quantifier free formulas Φ_V and Φ_f involving polynomials with integer coefficients. Denote by D_V and D_f the sum of the degrees of the polynomials in the respective formula. Let $D = \max\{D_V, D_f\}$ and let l be the maximum absolute value of the coefficients involved in the formulas.*

There exists a universal constant $c \in N$ such that, under the above conditions, we have

$$|f(x)| \le l^{D^{c(n+1)}}(1+|x|)^{D^{c(n-1)}}$$

for all x belonging to V.

Proof of Theorem 1.2. In part one we use Lemma 2.2 with the simplex Δ as V and $f = 1/F$. In our case $D = \max\{d+1, n+1\}$ and in the simplex Δ we have $(1+|x|) \le 2$. Taking into account that l and D are bigger than 2 we obtain a bound for $1/F$ in Δ:

$$1/F \le l^{D^{c(n-1)}} 2^{D^{c(n-1)}} = l^{D^{O(n)}}.$$

This completes the proof of part one. For part two we first note that the inequality $F_-(X+d) \le F_-(X) + d\sum(\partial F_-/\partial x_i)(X)$ is valid in the non-negative orthant. This follows from Taylor's multivariate theorem taking into account that F_- has only positive coefficients. As a consequence, the semialgebraic region G defined in Lemma 2.1 is contained in

$$G' = \left\{X : F(X) - d\sum \frac{\partial F_-}{\partial x_i}(X) \le 0,\ X \ge 0\right\}.$$

Notice that $F(X) - d\sum(\partial F_-/\partial x_i)(X) \le 0$ if and only if $\sum X \le d(\sum X)(\sum(\partial F_-/\partial x_i)(X))/F(X)$. The right-hand side of the last inequality is a quotient of two homogeneous polynomials of the same degree and we can bound it by the quotient of the maximum of $d(\sum X)(\sum(\partial F_-/\partial x_i)(X))$ and the minimum of $F(X)$ in the simplex. The minimum of $F(X)$ equals λ and the maximum of the numerator can be seen to be bounded by lnd^2, because of the following chain of inequalities:

$$d\left(\sum X\right)\left(\sum \frac{\partial F_-}{\partial x_i}(X)\right) \le d^2 n F_-(X) \le d^2 n l.$$

We have used that $\sum X = 1$ because we are in the unit simplex and $(\partial F_-/\partial x_i)(X) \le dF_-(X)$ because F_- has only positive coefficients. Thus $\sum X$ is bounded by (lnd^2/λ) in G' as desired. This completes the proof. □

Lemma 2.1 provides us with an algorithm to find a reasonably good bound for the Pólya exponent which is a priori smaller than those given in Theorem 1.2. We need to find the maximum for the linear functional $\sum X$ in the region G which was defined using $F_*(X) = F_+(X) - F_-(X+d)$. The maximum will be attained at a boundary point $Q = (q_1, \dots, q_n)$ such that $F_*(Q) = 0$ and the partial derivatives of F_* with respect to non-zero entries are all equal. This allows us to use symbolic methods (such as Gröbner bases). Nevertheless, since we are only interested in an upper bound for the Pólya exponent, it is enough for our purposes to apply numerical optimization techniques (such as numerical Lagrange multipliers). In the following table we show the value of the maximum $\sum X$ in G for several polynomials and compare it with the Pólya exponent. The values in the last column have been found by means of Gröbner bases and real root isolation.

	Pólya exponent	$\lceil \max_{X \in G}(\sum X) \rceil$
$1000x^2 - 1999xy + 1000y^2$	3997	15994
$50x^2 - 99xy + 50y^2$	197	794
$(50x^2 - 99xy + 50y^2)(x^2 + y^2)$	193	3180
$(50x^2 - 99xy + 50y^2)(x^4 + x^2y^2 + y^4)$	187	7158
$(x-y)^2(x+6y)^2 + y^4$	197	1948
$5x^4 + (x-y)^2(x+6y)^2 + y^4$	44	367
$10x^4 + (x-y)^2(x+6y)^2 + y^4$	30	228
$(x-z)^2 + (y-z)^2 + (x+y)^2$	3	19
$(412x^4 - 18x^3y + 556x^2y^2 + 40xy^3 + 533y^4 - 24x^3 - 344x^2y + 184xy^2 - 200y^3 + 540x^2 + 134xy + 678y^2 - 182x - 92y + 444)$	2	30

The last example in the table is a sum of the squares of 50 randomly generated quadratic forms, and will be used in Section 3 as an example of the process described in Theorem 3.2. The coefficients of the quadratic forms were generated using MAPLE's random numbers subroutine with $[-5, \dots, 5]$ as the range of variation. Our computational experience indicates that such "random" polynomials tend to have a low Pólya exponent.

Let us analyze in detail an example that contains as particular cases the first two polynomials in the table. Consider $F(X) = x_1^n + \cdots + x_n^n - (n-\varepsilon)x_1 \cdots x_n$ for a positive and small ε and large n. As pointed out in [5] its Pólya exponent is approximately $(n^3(n-1)/2\varepsilon)$. The maximum of $\sum X$ in G is attained at a point with $x_0 = \cdots = x_n$ and it is approximately (n^4/ε). So, the bound given by Lemma 2.1 equals the Pólya exponent asymptotically up to a factor of 2.

We can deduce some important consequences of this example: Pólya's theorem is not true if F is only non-negative [5] or if it is strictly positive only in the *open* orthant (e.g. $F(x,y,z) = (x-y)^2 + z^2$). The theorem is again not true over non-Archimedean fields (taking ε to be an infinitesimal). Finally, the Pólya exponent p cannot be bounded only by the degree d and the number of variables n of F (for these last two comments see [9]). Any bound will necessarily include the size l of its coefficients.

3. Decomposition of strictly positive polynomials

In this section we will connect Pólya's theorem to Hilbert's 17th problem. This problem asked whether every non-negative real polynomial can be expressed as a quotient of sums of squares of real polynomials. It was non-constructively solved by Artin in 1928 and other solutions have been proposed later, which are constructive or give conditions and bounds on the output polynomials. We recommend [1] and [2] for a brief history of the problem ([2] puts special emphasis on constructive aspects of the solution).

Pólya's theorem was used by Habicht [4] to give explicit solutions to Hilbert's 17th problem in the case of positive definite homogeneous polynomials. Here, we present a different way to do this. If we have a positive definite homogeneous polynomial F in n variables, Pólya's theorem can be applied to $F(\varepsilon_1 x_1, \varepsilon_2 x_2, \dots, \varepsilon_n x_n)$, where $\varepsilon_i \in \{+,-\}$. In this way we have 2^n Pólya-like-expressions, each of them certifying the positiveness of F inside a different orthant. We proceed to glue these local certificates with techniques similar to those in [6]. The decomposition of F obtained in this way is a quotient of two sums of even powers of monomials in the "variables" $x_1, x_2, \dots, x_n, F$. Let us remark that Reznick [8] has also given, using less elementary techniques, concrete decompositions for the same family of polynomials. His decomposition has a sum of even powers of linear forms in the numerator and a power of $\sum x_i^2$ in the denominator.

For convenience we will state all results in this section for an inhomogeneous polynomial F. This is possible provided that its homogenization is positive definite or, equivalently, if F is strictly positive and its largest degree part is positive definite. Reciprocally any positive definite homogeneous polynomial can be dehomogenized yielding an inhomogeneous polynomial with the above conditions. Hence Theorem 3.2 applies to homogeneous polynomials as well. In the following discussion $\boldsymbol{K}$ will denote any ordered field and $\boldsymbol{K}_+$ denotes the set of strictly positive elements in $\boldsymbol{K}$. Only in the last part of Theorem 3.2 we need $\boldsymbol{K}$ to be the rationals in order to apply the bound in part one of Theorem 1.2.

Lemma 3.1. *Let $F \in \boldsymbol{K}[x_1, \dots, x_n]$. Suppose that for a given x_i we have two identities $F \cdot A_1 = B_1$ and $F \cdot A_2 = B_2$ where A_1, B_1 are polynomials in $\boldsymbol{K}_+[x_i, T, F^2]$ and A_2, B_2 are polynomials in $\boldsymbol{K}_+[-x_i, T, F^2]$, for some arbitrary set of indeterminates T. Assume that both B_1 and B_2 have a non-zero constant term. Then we can find an expression of the form $F \cdot R = S$ where R and S are polynomials in $\boldsymbol{K}_+[x_i^2, T, F^2]$ and S has a non-zero constant term. Moreover $deg(S) \leq deg(B_1) + deg(B_2)$.*

Proof. We can decompose $A_1 = A_{1,1} + x_i A_{1,2}$, $B_1 = B_{1,1} + x_i B_{1,2}$, $A_2 = A_{2,1} - x_i A_{2,2}$, and $B_2 = B_{2,1} - x_i B_{2,2}$ with $A_{1,1}, A_{1,2}, B_{1,1}, B_{1,2}, A_{2,1}, A_{2,2}, B_{2,1}$ and $B_{2,2} \in \boldsymbol{K}_+[x_i^2, T, F^2]$. Separate the two identities in the form

$$FA_{1,1} - B_{1,1} = -x_i FA_{1,2} + x_i B_{1,2}, \quad FA_{2,1} - B_{2,1} = x_i FA_{2,2} - x_i B_{2,2}.$$

Multiplying side by side the above equations and grouping together terms with F, we obtain

$$F \cdot (A_{1,1}B_{2,1} + B_{1,1}A_{2,1} + x_i^2 A_{1,2}B_{2,2} + x_i^2 B_{1,2}A_{2,2})$$
$$= F^2(A_{1,1}A_{2,1} + x_i^2 A_{1,2}A_{2,2}) + B_{1,1}B_{2,1} + x_i^2 B_{1,2}B_{2,2}.$$

By hypothesis both $B_{1,1}$ and $B_{2,1}$ have a non-zero constant term and thus $B_{1,1}B_{2,1}$ has a non-zero constant term. The constant term of $F^2A_{1,1}A_{2,1}$ is either zero or positive, and thus the constant term of the right-hand side of the equation above is positive. From the above expression it is clear that $deg(S) \leq deg(B_1) + deg(B_2)$. □

As an immediate application of the above lemma and as a preparation for the multivariate case we present a method to decompose a real univariate strictly positive polynomial F as a quotient of two sums of squares. We remark that in the univariate case, the additional condition of F having a positive definite largest-degree part is redundant. Applying Theorem 1.2 to the homogenization of F we have the following expression where $B_1(x)$ has only positive coefficients

$$F(x)(x+1)^p = B_1(x).$$

With the same process applied to the polynomial $F(-x)$ we obtain

$$F(x)(1-x)^q = B_2(-x).$$

Taking $A_1 = (x+1)^p$, $A_2 = (1-x)^q$ and $T = \emptyset$ we are in the situation of Lemma 3.1. This will give an expression $F \cdot R = S$ with R, S polynomials in $\mathbb{R}_+[x^2, F^2]$ and thus sums of squares.

Theorem 3.2. *Let $F(x_1, x_2, \ldots, x_n)$ be a real strictly positive polynomial of degree d, whose homogenization is positive definite. For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ in $E^n = \{+,-\}^n$, let p_ε be the Pólya exponent of the homogenization of F in the orthant where the sign of the ith coordinate equals ε_i. Let $P = \sum_{\varepsilon \in E^n} p_\varepsilon$ and D be the maximum of $d+1$ and $n+1$. Then we can write*

$$F \cdot R = S,$$

where $R, S \in \mathbb{R}_+[x_1^2, x_2^2, \ldots, x_n^2, F^2]$ and $deg(S) \leq P + 2^n d$ (where S is considered as a polynomial in the original variables $x_1, x_2, \ldots, x_n$ to compute $deg(S)$).

If $F \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, then we can find R and S in $\mathbb{Z}_+[x_1^2, x_2^2, \ldots, x_n^2, F^2]$. We can also choose R and S with $l^{D^{O(n)}}$ monomials, where l is an upper bound for the absolute values of the coefficients of F.

Proof. Let $E^n = \{+,-\}^n$. For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E^n$ we have a Pólya expression in the corresponding orthant

$$F(x_1, x_2, \ldots, x_n) \cdot A_\varepsilon = B_\varepsilon,$$

where $A_\varepsilon, B_\varepsilon \in R_+[\varepsilon_1 x_1, \dots, \varepsilon_n x_n]$. Moreover, $A_\varepsilon = (1 + \varepsilon_1 x_1 + \cdots + \varepsilon_n x_n)^{p_\varepsilon}$ and thus B_ε has degree $p_\varepsilon + d$ and a non-zero constant term. Our goal is to glue the 2^n expressions in pairs using Lemma 3.1. More explicitly, for each $\sigma \in E^{n-1}$ consider the two expressions $FA_\varepsilon = B_\varepsilon$ and $FA_{\varepsilon'} = B_{\varepsilon'}$ where $\varepsilon = (\sigma, +)$ and $\varepsilon' = (\sigma, -)$.

We can apply Lemma 3.1 with $T = \{\sigma_1 x_1, \sigma_2 x_2, \dots, \sigma_{n-1} x_{n-1}\}$. This will give $2^{(n-1)}$ expressions (one for each $\sigma = (\sigma_1, \dots, \sigma_{n-1})$ in E^{n-1}) where the variable x_n always appears squared. Inductively, for each $\tau \in E^{n-2}$, we take the two expressions $FA_\sigma = B_\sigma$ and $FA_{\sigma'} = B_{\sigma'}$ with $\sigma = (\tau, +)$ and $\sigma' = (\tau, -)$ and apply Lemma 3.1 with $T = \{\tau_1 x_1, \tau_2 x_2, \dots, \tau_{n-2} x_{n-2}, x_n^2\}$. This process can be continued until all the variables appear squared.

For the degrees we note that in each gluing the degrees of the expressions glued are added. The degree of the final expression will be the sum of the degrees of the 2^n equations derived from Theorem 1.2. This gives the bound $P + d2^n$. □

We want to illustrate our method with a simple example. Consider the last polynomial given as an example in Section 2: $F := 134xy - 92y - 182x - 24x^3 + 412x^4 + 540x^2 + 678y^2 + 533y^4 - 200y^3 - 18x^3y + 556x^2y^2 - 344x^2y + 40xy^3 + 184xy^2 + 444$.

We will apply the process described in the proof of Theorem 3.2. Pólya's theorem applied to F in each one of the four orthants gives the following four identities (we show the intermediate distributions of the terms with respect to parity of the powers of y):

(i) $F(1+x+y)^2 = F(1+2x+x^2+y^2+y(2+2x)) = 1722x^2y^2 + 1442xy^2 + 874x^3 + 904x^4 + 706x + 620x^2 + 938y^2 + 811y^4 + 444 + 548x^3y^2 + 932x^4y^2 + 930y^4x + 1169y^4x^2 + 800x^5 + 412x^6 + 533y^6 + y(474x + 460x^3 + 548x^2 + 1498xy^2 + 796 + 1064y^2 + 396x^4 + 806x^5 + 1106y^4x + 1016x^2y^2 + 1134x^3y^2 + 866y^4)$

(ii) $F(1+x-y)^2 = F(1+2x+x^2+y^2-y(2+2x)) = 2562x^2y^2 + 1274xy^2 + 874x^3 + 904x^4 + 706x + 620x^2 + 1306y^2 + 1611y^4 + 444 + 1996x^3y^2 + 1004x^4y^2 + 1570y^4x + 1009y^4x^2 + 800x^5 + 412x^6 + 533y^6 - y(574x + 1604x^3 + 884x^2 + 1950xy^2 + 980 + 1648y^2 + 1156x^4 + 842x^5 + 1026y^4x + 1944x^2y^2 + 1090x^3y^2 + 1266y^4)$.

(iii) $F(1-x+y)^2 = F(1-2x+x^2+y^2+y(2-2x)) = 450x^2y^2 - 902xy^2 - 1286x^3 + 1000x^4 - 1070x + 1348x^2 + 938y^2 + 811y^4 + 444 - 300x^3y^2 + 1004x^4y^2 - 402y^4x + 1009y^4x^2 - 848x^5 + 412x^6 + 533y^6 + y(-934x - 324x^3 + 740x^2 - 414xy^2 + 796 + 1064y^2 + 564x^4 - 842x^5 + 866y^4 - 1026y^4x + 120x^2y^2 - 1090x^3y^2)$.

(iv) $F(1-x-y) = 716x^2y^2 - 628xy^2 - 564x^3 + 436x^4 - 626x + 722x^2 + 770y^2 + 733y^4 + 444 - 538x^3y^2 - 573y^4x - 412x^5 - y(-408x - 350x^3 + 1018x^2 - 56xy^2 + 536 + 878y^2 + 394x^4 + 533y^4 + 596x^2y^2)$.

Applying Lemma 3.1, with y as the distinguished variable to the pairs (i)–(ii) and (iii)–(iv), and grouping terms as in Lemma 3.1 we get (notice the expressions are presented now arranged by parity of powers of the variable x):

(i)–(ii): $F(7508x^2y^6 + 18160x^2y^2 + 6544x^4 + 4952x^2 + 6684y^2 + 10090y^4 + 20348x^4y^2 + 26700y^4x^2 + 5832x^6 + 7752y^6 + 8562x^4y^4 + 6056x^6y^2 + 824x^8 + 1066y^8 + 888 + x(14264y^2 + 5640x^2 + 3188 + 22568x^2y^2 + 22380y^4 + 6964x^4 + 14416x^4y^2 + 19768y^4x^2 + 13160y^6 + 3248x^6))$

$= 20679124x^2y^6 + 6723652x^2y^2 + 2421240x^4 + 1048996x^2 + 1776416y^2 + 4654924y^4 + 12531112x^4y^2 + 16893804y^4x^2 + 3380292x^6 + 6653476y^6 + 24185356x^4y^4 + 12976984x^6y^2 + 3666432x^8y^4 + 5547524x^6y^6 + 1476284x^{10}y^2 + 4580433x^4y^8 + 2295630y^{10}x^2 + 169744x^{12} + 284089y^{12} + 6F^2x^2y^2 + F^2 + 6F^2x^2 + 6F^2y^2 + F^2x^4 + F^2y^4 + 17274928x^6y^4 + 20296116x^4y^6 + 7688272x^8y^2 + 13108438x^2y^8 + 2726496x^8 + 5276765y^8 + 1384896x^{10} + 2387282y^{10} + 197136 + x(3711592y^2 + 1651552x^2 + 626928 + 10260368x^2y^2 + 10310324y^4 + 3070608x^4 + 14459768x^4y^2 + 22021832y^4x^2 + 14337360y^6 + 3621212y^{10} + 9347744x^6y^4 + 12882352x^4y^6 + 3862096x^8y^2 + 9701716x^2y^8 + 659200x^{10} + 12F^2y^2 + 2166576x^8 + 4F^2 + 4F^2x^2 + 22429140x^4y^4 + 22897032x^2y^6 + 10878896x^6y^2 + 10718648y^8 + 3153936x^6)$.

(iii)–(iv): $F(8408\,x^2y^2 + 4572\,x^4 + 4836\,x^2 + 4020\,y^2 + 5134\,y^4 + 5584\,x^4y^2 + 5430\,y^4x^2 + 2520\,x^6 +$
$3198\,y^6 + 888 - x(7456\,y^2 + 5268\,x^2 + 3028 + 6972\,x^2y^2 + 6162\,y^4 + 3696\,x^4 + 3584\,x^4y^2 + 4402\,y^4x^2 +$
$3198\,y^6 + 824\,x^6)$
$= 5298888\,x^2y^6 + 5057552\,x^2y^2 + 3019356\,x^4 + 1588900\,x^2 + 1185008\,y^2 + 2676988\,y^4 + 6822948\,x^4y^2 +$
$6963772\,y^4x^2 + 3189648\,x^6 + 3371312\,y^6 + 6633028\,x^4y^4 + 5042212\,x^6y^2 + F^2 + 3\,F^2x^2 + 3\,F^2y^2 + 3085788x^6y^4 +$
$3212416\,x^4y^6 + 1829476\,x^8y^2 + 1989123\,x^2y^8 + 1741568\,x^8 + 2332333\,y^8 + 529008\,x^{10} + 852267\,y^{10} +$
$197136 - x(2915800\,y^2 + 2437788\,x^2 + 753024 + 6141928\,x^2y^2 + 4529144\,y^4 + 3340724\,x^4 + 6443844\,x^4y^2 +$
$7068028\,y^4x^2 + 4175308\,y^6 + 852267\,y^{10} + 2123228\,x^6y^4 + 2840400\,x^4y^6 + 967052\,x^8y^2 + 2057377\,x^2y^8 +$
$169744\,x^{10} + 3\,F^2y^2 + 1014096\,x^8 + 3\,F^2 + F^2x^2 + 5639176\,x^4y^4 + 4866908\,x^2y^6 + 3468572\,x^6y^2 + 2264079\,y^8 +$
$2550240\,x^6)$.

Finally, applying again Lemma 3.1 with x as the distinguished variable, we get the following expression from which F is decomposed as a quotient of two sums of squares:

$F(716381628672\,x^2y^6 + 80945077792\,x^2y^2 + 1309788013600\,x^{10}y^4 40286070144\,x^4 + 8570997312\,x^2 +$
$4739888256\,y^2 + 24573722112\,y^4 + 318423130400\,x^4y^2 + 317633301040\,y^4x^2 + 99294039872\,x^6 +$
$68743283680\,y^6 + 1060688074256\,x^4y^4 + 672316077216\,x^6y^2 + 143469890432\,x^{14}y^2 +$
$1982702974736\,x^8y^4 + 2814822717360\,x^6y^6 + 768832371904\,x^{10}y^2 + 2303389497104\,x^4y^8 +$
$990053073384\,y^{10}x^2 + 125406307008\,x^{12} + 116435744860\,y^{12} + 1863486418736\,x^6y^4 +$
$1983138377456\,x^4y^6 + 886558335136\,x^8y^2 + 1032502294416\,x^2y^8 + 505612381472\,x^{12}y^4 +$
$157752051648\,x^8 + 122024198048\,y^8 + 169540742272\,x^{10} + 144755788904\,y^{10} + 40199723460\,x^2y^{16} +$
$2507919676288\,x^6y^8 + 1687333237608\,x^4y^{10} + 2293828274592\,x^8y^6 + 621063650084\,x^2y^{12} +$
$1282298534376\,x^6y^{10} + 1423499576064\,x^8y^8 + 379456896112\,x^8y^{10} + 292967217500\,x^6y^{12} +$
$338955778976\,x^{10}y^8 + 145245087972\,x^4y^{14} + 1051819997200\,x^{10}y^6 + 235854251632\,x^2y^{14} +$
$208393538448\,x^{12}y^6 + 60652193280\,x^{14} + 735231433668\,x^4y^{12} + 1817033244\,y^{18} + 20512201152\,x^{16}y^2 +$
$60546876076\,y^{14} + 17483777024\,x^{16} + 18186081524\,y^{16} + 1958166784\,x^{18} + 350113536 + (39456\,x^2 +$
$109168\,x^4 + 60284\,y^4 + 128196\,x^2y^8 + 89728\,x^8y^2 + 76044\,y^6 + 227652\,x^4y^6 + 395200\,x^2y^6 + 48644\,y^8 +$
$11536\,x^{10} + 6396\,y^{10} + 208940\,x^6y^4 + 551252\,x^4y^4 + 342344\,x^6y^2 + 451140\,y^4x^2 + 430776\,x^4y^2 +$
$238456\,x^2y^2 + 1776 + 72800\,x^8 + 18696\,y^2 + 122640\,x^6)F^2 + 85194907968\,x^{14}y^4 + 430307879168\,x^{12}y^2)$
$= (153448078017504\,x^2y^6 + 11011029982720\,x^2y^2 + 851952864568912\,x^{10}y^4 + 5511262925968\,x^4 +$
$992116096128\,x^2 + 583803281664\,y^2 + 3550450975360\,y^4 + 53477285280880\,x^4y^2 + 53285071815840\,y^4x^2 +$
$16742340875856\,x^6 + 12247813101568\,y^6 + 228076657290496\,x^4y^4 + 144275999563072\,x^6y^2 +$
$177419113096224\,x^{14}y^2 + 818449513424880\,x^8y^4 + 1160297022426176\,x^6y^6 + 316404880090816\,x^{10}y^2 +$
$943080312658456\,x^4y^8 + 400790099977488\,y^{10}x^2 + 51946321935360\,x^{12} + 51812546094456\,y^{12} +$
$537983815619840\,x^6y^4 + 571695232269920\,x^4y^6 + 255064333134736\,x^8y^2 + 295172928849576\,x^2y^8 +$
$613138233076304\,x^{12}y^4 + 33884952842288\,x^8 + 27834477697840\,y^8 + 48891319043696\,x^{10} +$
$44539310679888\,y^{10} + 132667868133349\,x^2y^{16} + 1609709073374408\,x^6y^8 + 1074117047816472\,x^4y^{10} +$
$1484929624125296\,x^8y^6 + 391920773628424\,x^2y^{12} + 1494743643985512\,x^6y^{10} + 1679981404535176\,x^8y^8 +$
$1210562461740080\,x^8y^{10} + 922584991406952\,x^6y^{12} + 1101811041447776\,x^{10}y^8 + 463071536349080\,x^4y^{14} +$
$1257591653271104\,x^{10}y^6 + 274626206521816\,x^2y^{14} + 697775379415696\,x^{12}y^6 + 40598074878080\,x^{14} +$
$854507981216656\,x^4y^{12} + 11022901919929\,y^{18} + 76241258273152\,x^{16}y^2 + 44002353922224\,y^{14} +$
$22913533222784\,x^{16} + 26786486470753\,y^{16} + 8809248338688\,x^{18} + 296435588380336\,x^{14}y^4 +$
$40113876079218\,y^{18}x^2 + 5607817144761\,y^{20}x^2 + 25101305044221\,x^4y^{18} + 20172563339904\,x^{18}y^2 +$
$2384556922240\,x^{20}y^2 + 63315154652095\,x^6y^{16} + 128071716208936\,x^{10}y^{12} + 11885395021040\,x^{18}y^4 +$
$106193345637816\,x^8y^{14} + 75244284164400\,x^{14}y^8 + 113979811644760\,x^{12}y^{10} + 36042080253584\,x^{16}y^6 +$
$156928941649179\,y^{16}x^4 + 87953191492976\,x^{16}y^4 + 351913378400864\,x^6y^{14} + 234657785839200\,x^{14}y^6 +$
$525673478728032\,x^8y^{12} + 427778646436656\,x^{12}y^8 + 557728278696952\,x^{10}y^{10} + 38862602496 +$
$281083319515232\,x^{12}y^2 + 2697191817931\,y^{20} + 242119679763\,y^{22} + 2064497141504\,x^{20} + 201691178752\,x^{22} +$
$(3\,y^6 + 45\,x^4y^2 + 19\,y^4 + 21\,x^2 + 9\,y^2 + 1 + 90\,x^2y^2 + 57\,y^4x^2 + 35\,x^4 + 7\,x^6)F^4 + (1011238152\,x^2y^6 +$
$214583120\,x^2y^2 + 213172104\,x^{10}y^4 + 104354136\,x^4 + 27650008\,x^2 + 14240800\,y^2 + 60356936\,y^4 +$
$628009464\,x^4y^2 + 633987416\,y^4x^2 + 191801384\,x^6 + 125837552\,y^6 + 1474515760\,x^4y^4 + 929987432\,x^6y^2 +$
$932892160\,x^8y^4 + 1345813968\,x^6y^6 + 355155544\,x^{10}y^2 + 1110516186\,x^4y^8 + 486373732\,y^{10}x^2 +$
$54326496\,x^{12} + 45155610\,y^{12} + 1635596488\,x^6y^4 + 1755400200\,x^4y^6 + 771165424\,x^8y^2 + 932373358\,x^2y^8 +$
$214189576\,x^8 + 155761526\,y^8 + 142279872\,x^{10} + 112787062\,y^{10} + 436152642\,x^6y^8 + 292702206\,x^4y^{10} +$
$386752696\,x^8y^6 + 107817366\,x^2y^{12} + 7129248\,x^{14} + 5113602\,y^{14} + 1182816 + 65068392\,x^{12}y^2)F^2)$

Acknowledgements

We thank L. González-Vega, L.M. Pardo and B. Sturmfels for many useful comments and conversations. We also thank the referee for suggestions that improved the presentation.

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